AD-A037 556 BROWN UNIV PROVIDENCE R I DIV OF ENGINEERING F/G 16/2
MINIMUM ENERGY CONTROL OF A BILINEAR PURSUIT-EVASION SYSTEM.(U)
JUL 76 K WEI, A E PEARSON AF-AFOSR-2793-75
UNCLASSIFIED AFOSR-TR-77-0174 NL

| OF |
AD
A037556

END
DATE
FILMED
4-77

MINIMUM ENERGY CONTROL OF A BILINEAR PURSUIT-EVASION SYSTEM*

by

Kuang-Chung Wei and Allan E. Pearson

Division of Engineering
and
Lefschetz Center for Dynamical Systems
Brown University
Providence, Rhode Island 02912

July 21, 1976

*This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR-75-2793.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR - TR - 77 - 0174
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): MINIMUM ENERGY CONTROL OF A BILINEAR PURSUIT-EVASION SYSTEM
5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Kuang-Chung Wei, Allan E. Pearson
8. CONTRACT OR GRANT NUMBER(s): AFOSR -2793-75
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Division of Engineering, Brown University, Providence, Rhode Island 02912
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F, 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB DC 20332
12. REPORT DATE: July 21, 1976
13. NUMBER OF PAGES: 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

065 310

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A two-dimensional missile intercept system is formulated as a bilinear pursuit-evasion problem. The minimum energy control of this problem is discussed and analyzed together with the singularly perturbed problem. It is shown that the optimal control appears in a constant form which can be solved explicitly from the boundary conditions. The terminal intercept condition can be achieved provided the area swept out by a given contrdler sustains a certain constant value. A least squares estimation scheme for the target speed and the relative

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20. Abstract

heading is developed for the case where the target turn rate is a known constant.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

ABSTRACT

A two-dimensional missile intercept system is formulated as a bilinear pursuit-evasion problem. The minimum energy control of this problem is discussed and analyzed together with the singularly perturbed problem. It is shown that the optimal control appears in a constant form which can be solved explicitly from the boundary conditions. The terminal intercept condition can be achieved provided the area swept out by a given controller sustains a certain constant value. A least squares estimation scheme for the target speed and the relative heading is developed for the case where the target turn rate is a known constant.

A

I. Introduction

This paper is devoted to the application of the bilinear regulator theory, developed recently by the authors [1], to a two-dimensional pursuit-evasion problem. The general pursuit-evasion problem has been studied for many years resulting in a variety of formulations and control schemes. Among these, Deyst [2] applied the theory of optimal stochastic control to derive a sophisticated nonlinear controller which lead to a significant improvement in performance in relation to those resulting from the conventional proportional navigation guidance law. On the other hand, the theory of differential games was considered by Ho *et al* [3] and Rajan [4] to determine optimal strategies for this problem. Recently, Slater [5] and Nazaroff [6] took the perturbation point of view in applying optimal control theory to this two-dimensional pursuit-evasion problem.

However, due to the highly nonlinear (transcendental type) structure underlying the system, the usual linearization techniques are not sufficient to render an accurate maneuver of the pursuer under a variety of operational circumstances [5]. On the other hand, the currently developed numerical approaches in solving this terminal constraint optimal control problem generally require a tremendous amount of storage space in addition to a significant computing time. Therefore, a direct approach using a bilinear control formulation is considered here which takes advantage of the merits of a commutative bilinear system [1] to analyze the nonlinear optimization problem directly. A closed form solution to this problem

is obtained. It provides a great deal of potential to be implemented in practice due to the structural constancy of the controller.

The commutative property of such a bilinear system is introduced by an additional assumption, beyond the traditional control configuration, that the line speed of the pursuer (missile) can be manipulated proportional to its turn rate which is regarded as the major control variable. The line speed and turn rate of the target are assumed to be known constants. For a given initial position of the target and relative heading with respect to the missile, the problem is to determine a proportional factor γ, a proper intercept time T and a control law (turn rate of the missile) which steers the missile to the target at T; meanwhile, the total control energy spent during this interval is minimized. The existence theorems for the bilinear regulator problem [1] are applied to show the simple form of the optimal control law which is a constant and can be solved explicitly from the boundary conditions. It is thus shown that the missile is globally controllable to the target with the control law formulated here provided the turn rate of the target does not vanish.

A singular perturbation of this system is also considered. Singularly perturbed optimal control problems have been studied in Reference [7] where the asymptotic approximation approach is considered in obtaining asymptotic solutions for either linear or nonlinear systems. Here we will derive the optimal control in closed form for this problem and show that this solution converges to that of the unperturbed system as the degree of perturbation ε approaches zero.

A least squares estimation scheme is developed to estimate the target speed and relative heading based on the known target and missile turn rates and continuous measurements of the position vector. This scheme in combination with the optimal control law is simulated under a variety of initial conditions and target line speeds and turn rates to show the effectiveness of such an optimal controller.

The problems of further interest include the construction of a feedback control law which can guarantee a terminal intercept, the sensitivity analysis as well as the stochastic estimation of the target speed and turn rate which should be taken into account for the implementation of a desirable controller in a real pursuit-evasion system.

II. Minimum Energy Control of Commutative Bilinear Systems

Consider a general bilinear system

$$\dot{x} = (A + \sum_{i=1}^{m} B_i u_i)x, \qquad x(t_0) = x_0 \qquad t \in [t_0,T] \tag{1}$$

where $A, B_i (i=1,\ldots,m)$ are constant $n \times n$ matrices; $u(t)$ is an m-vector with each component u_i being a square integrable function defined on $[t_0,T]$. The system (1) is commutative if every pair of the matrices $(A,B_1,B_2,\ldots,B_m)$ commute with each other.

Due to its structural simplicity, we will focus our attention on the commutative bilinear system and the minimum energy problem associated with it in this paper.

The objective is to minimize the following cost functional

$$J(u) = \int_{t_0}^{T} u'(t)Ru(t)dt \qquad (2)$$

subject to the constraint

$$x(T) = x_1 \qquad (3)$$

where R is an $m \times m$ positive definite symmetric matrix; and x_1 is a prespecified vector.

In order to analyze this terminal constraint problem, we shall first study the reachable zone of the system (1). We introduce the following definitions. Let U be the set of admissible controls.

Definition 1:

A set $Z(x_0;U)$ is called a reachable set associated with (1) if $Z(x_0;U) = \{x(T)\varepsilon R^n : x(t_0) = x_0, u(t)\varepsilon U, x(t)$ satisfies (1) in $[t_0,T]\}$. A set $Z(D;U)$ is called a reachable zone associated with (1) if $Z(D;U) = \bigcup_{x_0 \varepsilon D} Z(x_0;U)$.

Definition 2:

The system (1) is reachable to x_1 with respect to x_0 and U if there is an input $u(t) \varepsilon U$ which steers it from x_0 to x_1 at some finite time T. It is constant reachable to x_1 with respect to x_0 if there is a constant input function u_c which steers it from x_0 to x_1 at some finite time T.

The next lemma reveals an interesting property regarding the reachability of commutative bilinear systems. The proof is straightforward and may be found in Reference [1] or [8].

Lemma 1:

The commutative bilinear system (1) is reachable to x_1 with respect to x_0 and $L^2([t_0,T],R^m)$ if and only if it is constant reachable to x_1 with respect to x_0.

Lemma 1 assures that if $u(t) \varepsilon L^2([t_0,T],R^m)$ steers the commutative bilinear system from x_0 to x_1 at T, then there exists a constant input function u_c which can do the same job as well. This enables us to study the given system with a class of simple easily-implemented input functions, namely, the constant input functions. For this class of bilinear systems, the reachable zone is much easier to characterize because one need only consider a constant input u_c as a set of parameters, then compute the corresponding transition matrices† $\Phi_{c_i}(t,t_0)$, $i = 1,\ldots,m$, which characterize the reachable zone of the given system (1) independent of the initial conditions. The following theorem describes the existence and the form of optimal controls to the problem (1) - (3). Again, the interested reader is referred to Reference [1] or [8] for the proof.

---

† The transition matrix $\Phi_{c_i}(t,t_0)$ is defined by $\dot{\Phi}_{c_i}(t,t_0) = B_i u_i \Phi_{c_i}(t,t_0)$ with $\Phi_{c_i}(t_0,t_0) = I$, the identity matrix.

Theorem 1:

Given a commutative bilinear system (1) with the cost (2). If x_1 belongs to the reachable set $Z(x_0;L^2([t_0,T],R^m))$, then there exists a constant optimal control u_c^* which steers (1) from x_0 to x_1 at T and minimizes (2). Furthermore, u_c^* satisfies

$$x_1 = \Phi_A(T,t_0) \prod_{i=1}^{m} \Phi_{c_i}(T,t_0)x_0. \tag{4}$$

Therefore, the infinite dimensional optimization problem (1) - (3) has been transformed into solving a finite dimensional non-linear algebraic equation (4) for u_c^*. It will be shown in the next section that in the pursuit-evasion problem we can easily solve for the optimal control in analytic form. This is certainly due to the structural merit of a commutative bilinear system.

III. A Two-Dimensional Missile Intercept System

For a typical high-speed pursuing missile and short initial range, it is assumed that the maneuvering of the vehicles can be restricted to a two-dimensional plane as shown in Figure 1 with the coordinates fixed in the missile.

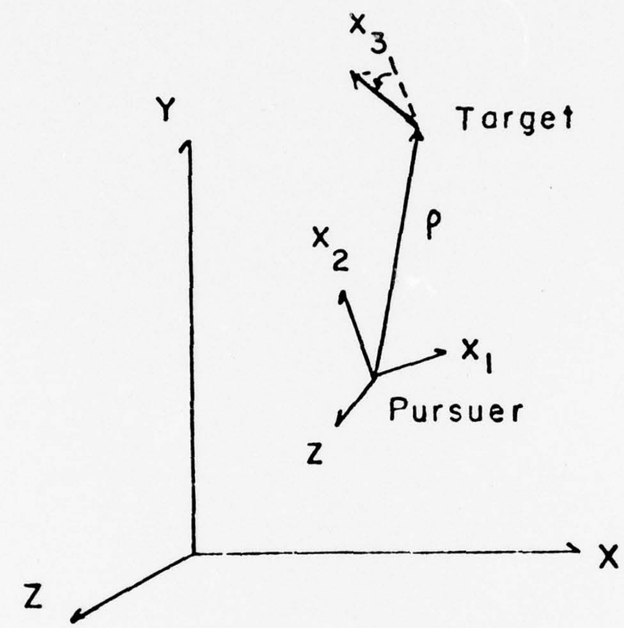


Fig. 1 Dynamics of a Two-Dimensional Pursuit System

Kinematic Equations of Motion [5].

Denote the angular rate of the missile and the target with respect to a non-rotating reference frame as u_p and u_T, respectively. Then the equations of motion are described by:

$$\dot{x}_1 = -v_T \sin x_3 + x_2 u_p$$

$$\dot{x}_2 = v_T \cos x_3 - x_1 u_p - v_p \qquad (5)$$

$$\dot{x}_3 = u_T - u_p$$

where v_T and v_p are the line speeds of the target and the missile relative to air, x_1 and x_2 are the horizontal and vertical distance from the missile, and x_3 is the relative angle between the headings of the missile and target measured counterclockwise.

The system (5) can be transformed into a homogeneous bilinear system by introducing the auxiliary states: $x_4 = \sin x_3$, $x_5 = \cos x_3$ and $x_6 = 1$. That is,

$$\dot{x} = Ax + Bxu \qquad (6)$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & -v_p \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad x(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ \sin x_3(t_0) \\ \cos x_3(t_0) \\ 1 \end{pmatrix} \qquad (7)$$

in which $u = u_p$ is defined as the control variable.

The problem is to find a square integrable function $u^*(t)$ which steers the missile to the target at some finite time T (a free time formulation) while the total control energy consumed over this interval $[0,T]$ is minimized. Hence, the following cost functional is considered:

$$J(u) = \int_{t_0}^{T} u^2(t)dt \quad , \quad T > t_0 \tag{8}$$

subject to

$$x_1(T) = x_2(T) = 0. \tag{9}$$

Distinct from the classical proportional navigation guidance law, a crucial assumption is proposed relative to the line speed v_p of the missile which leads to an explicit solution to the aforementioned problem. That is, v_p is modeled to be proportional to (an amplification of) u_p with the proportional parameter γ to be determined by the boundary conditions. Hence, we assume

$$v_p = \gamma u_p \, . \tag{10}$$

With this condition, Equation (7) becomes

$$A = \begin{bmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{11}$$

In the event that v_T and u_T are constants, Equations (6) and (11) can be integrated explicitly as follows:

$$\begin{aligned} &x_6(t) = 1 \quad , \quad x_5(t) = \cos x_3(t) \quad , \quad x_4(t) = \sin x_3(t) \\ &x_3(t) = x_3(t_0) + u_T(t-t_0) - \int_{t_0}^{t} u(s)ds \end{aligned} \tag{12}$$

and

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} x_1(t_0)\cos \int_{t_0}^{t} u(s)ds + x_2(t_0)\sin \int_{t_0}^{t} u(s)ds \\ -x_1(t_0)\sin \int_{t_0}^{t} u(s)ds + x_2(t_0)\cos \int_{t_0}^{t} u(s)ds \end{bmatrix} \\ &+ v_T \int_{t_0}^{t} \begin{bmatrix} -\sin[x_3(t_0) - \int_{t_0}^{t} uds + u_T(\xi-t_0)] \\ \cos[x_3(t_0) - \int_{t_0}^{t} uds + u_T(\xi-t_0)] \end{bmatrix} d\xi + \gamma \begin{bmatrix} \cos \int_{t_0}^{t} uds - 1 \\ -\sin \int_{t_0}^{t} uds \end{bmatrix} . \end{aligned} \tag{13}$$

We will first resolve the terminal constraint problem by con-

sidering the intercept angle as a parameter, then incorporate the solutions with the minimum energy problem. Considerations should be given to two separate cases in which u_T is zero and non-zero, respectively.

<u>Non-zero Angular Maneuver of the Target</u> $(u_T \neq 0)$

The terminal constraint (9) on x_1 and x_2 requires (for some $T > t_0$):

$$0 = \begin{Bmatrix} [x_1(t_0)+\gamma]\cos[u_T(T-t_0)+x_3(t_0)-\beta] + x_2(t_0)\sin[u_T(T-t_0)+x_3(t_0)-\beta]-\gamma \\ -[x_1(t_0)+\gamma]\sin[u_T(T-t_0)+x_3(t_0)-\beta] + x_2(t_0)\cos[u_T(T-t_0)+x_3(t_0)-\beta] \end{Bmatrix}$$

$$+ \frac{v_T}{u_T}\begin{Bmatrix} \cos\beta - \cos[u_T(T-t_0)-\beta] \\ \sin[u_T(T-t_0)-\beta] + \sin\beta \end{Bmatrix} \tag{14}$$

where $\beta = x_3(t_0) - \int_{t_0}^{T} u(s)ds + u_T(T-t_0)$ is defined as the intercept angle.

Therefore, the terminal constraint problem has been reduced to solving a pair of nonlinear algebraic equations (14) for an appropriate set (γ,β,T). A solution often exists for this case in which the number of unknowns exceeds the number of equations. From Equation (14), we obtain [8]

$$\cos\beta = \cos x_3(t_0) - \frac{u_T}{v_T}x_1(t_0) + \frac{1}{\gamma}\Big\{x_1(t_0)\cos x_3(t_0)+x_2(t_0)\sin x_3(t_0) - \frac{u_T}{2v_T}[x_1^2(t_0)+x_2^2(t_0)]\Big\}. \tag{15}$$

From Equation (15), a proper γ and β can be solved in terms of the initial conditions $x_1(t_0), x_2(t_0), x_3(t_0), u_T$ and v_T. After γ and β are solved, the intercept time T can be computed by the following formula:

$$T = t_0 + \frac{1}{u_T}[2k\pi + \tan^{-1}\frac{F}{G}], \tag{16}$$

where

$$\begin{aligned} F &= \gamma[\gamma+x_1(t_0)]\sin[\beta-x_3(t_0)] + \gamma\, x_2(t_0)\cos[\beta-x_3(t_0)] \\ &\quad - \frac{v_T}{u_T}\{\gamma\sin\beta-[\gamma+x_1(t_0)]\sin x_3(t_0) + x_2(t_0)\cos x_3(t_0)\} \\ G &= \gamma[\gamma+x_1(t_0)]\cos[\beta-x_3(t_0)] - \gamma\, x_2(t_0)\sin[\beta-x_3(t_0)] \\ &\quad - \frac{v_T}{u_T}\{\gamma\cos\beta+[\gamma+x_1(t_0)]\cos x_3(t_0) + x_2(t_0)\sin x_3(t_0) - \frac{v_T}{u_T}\} \end{aligned}$$

$$k = 0, \pm 1, \ldots \quad \text{such that} \quad T > t_0.$$

These results are summarized in the next lemma.

<u>Lemma 2</u>:

If a constant but non-zero angular maneuver of the target is assumed, then there exists a triple (γ, β, T) satisfying (15) - (16) which solves the terminal constraint problem (6) and (9) for every $(x_1(t_0), x_2(t_0), x_3(t_0)) \in R^3$. The corresponding proper control action satisfies

$$\int_{t_0}^{T} u(s)ds = x_3(t_0) - \beta + u_T(T-t_0). \tag{17}$$

<u>Zero Angular Maneuver of the Target</u> ($u_T=0$)

In a similar manner, the terminal constraint becomes

$$0 = \begin{pmatrix} [\gamma+x_1(t_0)]\cos[\beta-x_3(t_0)]-x_2(t_0)\sin[\beta-x_3(t_0)]-v_T(T-t_0)\sin\beta - \gamma \\ \\ [\gamma+x_1(t_0)]\sin[\beta-x_3(t_0)]+x_2(t_0)\cos[\beta-x_3(t_0)]+v_T(T-t_0)\cos\beta \end{pmatrix} \tag{18}$$

where β is as defined in (14). This equation can be reduced to

$$\cos\beta = \cos x_3(t_0) + \frac{1}{\gamma}[x_1(t_0)\cos x_3(t_0) + x_2(t_0)\sin x_3(t_0)]. \tag{19}$$

Finally, the intercept time T is determined by

$$T = t_0 + \frac{1}{v_T}\{[\gamma+x_1(t_0)]\sin x_3(t_0) - x_2(t_0)\cos x_3(t_0) - \gamma\sin\beta\}. \tag{20}$$

A feasible T requires $T > t_0$, i.e., the term in the bracket must be positive. This is true only for those initial conditions outside the region [8] E defined by

$$E = \{(0,y,z) \in R^3\text{: either } y > 0,\ z = (2k+1)\pi, \text{ or } y < 0, z = 2k\pi;\ k = 0,\pm 1,\ldots \tag{21}$$

These results are summarized in the next lemma.

Lemma 3:

If a zero angular maneuver of the target is assumed, then there exists a triple (γ,β,T) which solves the terminal constraint problem (6) and (9) for every $(x_1(t_0),x_2(t_0),x_3(t_0)) \in R^3 \setminus E$.

It should be noticed that in the previous analysis the control function $u(t)$ has been eliminated for simplicity of computation. However, an admissible control which steers the missile to the target at T is associated with the triple (γ,β,T) through Equation (17). Thus the set U_c of admissible controls is specified by:

$$U_c = \left\{ u \epsilon L^2([t_0,T],R) : \int_{t_0}^{T} u(s)ds = x_3(t_0)-\beta+u_T(T-t_0) \right\} \tag{22}$$

where β and T are determined by Equations (15) - (16) or (19) - (20).

With the set U_c of admissible controls furnished as in (22), we are ready to derive the solution to the minimum energy problem (8). The following proposition is a direct consequence of Theorem 1.

Proposition 1:

Given the system (6) and (11), there exists an optimal control $u^* \epsilon U_c$ which minimizes the cost (8) subject to the constraint (9) for each appropriate set of initial conditions $(x_1(t_0),x_2(t_0),x_3(t_0),$ $u_T,v_T)$. This control is given by

$$u^*(t) = u_T + \frac{x_3(t_0)-\beta}{T-t_0} \tag{23}$$

where T and β are given as discussed in Lemmas 2 and 3.

IV. Singularly Perturbed Problem

A more complex system model is considered in this section in which the missile turn rate is taken as the output of a first-order lag. This takes into account the practical situation in which the missile turn rate is furnished by a DC-motor having first-order actuator dynamics. That is,

$$\varepsilon\, \dot{u}(t) = -u(t) + u_0 \qquad \varepsilon > 0 \tag{24}$$

where u_0 is the real input. In the limiting case where ε approaches zero, this consideration is generally known as a 'singular perturbation' problem [7].

The cost functional in this case becomes

$$J(u_0) = \frac{1}{2}\int_{t_0}^{T} u_0^2(s)\,ds \tag{25}$$

subject to the constraint

$$x_1(T) = x_2(T) = 0 \quad \text{for some} \quad T > t_0. \tag{26}$$

By defining $z = u$, the system equations can be expressed by:

$$\begin{aligned} \dot{x} &= Ax + Bxz \\ \varepsilon\dot{z} &= -z + u_0 \end{aligned} \qquad \varepsilon > 0 \tag{27}$$

where A and B are the same as defined in (7).

Because it was shown in the last section that the terminal constraint problem has a solution in which the control action satisfies (17), or in terms of the new state z

$$\int_{t_0}^{T} z(s)ds = x_3(t_0) - \beta + u_T(T-t_0),$$

the terminal constraint problem (26) also has a solution provided:

$$\frac{1}{\varepsilon}\int_{t_o}^{T}\int_{t_o}^{t} u_0(s)e^{-\frac{t-s}{\varepsilon}}\,ds\,dt = x_3(t_0)-\beta+u_T(T-t_0)+\varepsilon z(t_0)\left[e^{-\frac{T-t_0}{\varepsilon}} - 1\right]. \tag{28}$$

Thus the set U_c of admissible controls for the problem (25) - (27) is the collection of inputs satisfying (28).

The existence of an optimal control $u_0^* \varepsilon U_c$ which minimizes the cost (25) can be easily established as shown in Theorem 1. By the Maximum Principle, this optimal solution satisfies

$$u_0^*(t) = q(t) \tag{29}$$

and

$$\begin{aligned} \dot{p}(t) &= -\partial H/\partial x = -(A'+B'z)p \\ \varepsilon\dot{q}(t) &= -\partial H/\partial z = -x'B'p+q \end{aligned}\quad , \ \varepsilon q(T) = 0. \tag{30}$$

In the above expressions, (p,q) are costates corresponding to (x,z)

and prime denotes the matrix transpose operation. It can be easily checked that $\frac{d}{dt}(x'B'p) = 0$,
hence $x'B'p = k$, a constant to be determined by the boundary conditions.

Substituting k into (30), q can be solved:

$$q(t,\varepsilon) = -k\left[e^{-\frac{t-T}{\varepsilon}} - 1\right] = u_0^*(t). \tag{31}$$

Then from Equations (28) - (29), k is given by:

$$k = \frac{1}{L}\left[x_3(t_0) - \beta + u_T(T-t_0) + \varepsilon z(t_0)(e^{-\frac{T-t_0}{\varepsilon}} - 1)\right] \tag{32}$$

where

$$L = \varepsilon\left[-1.5 + \frac{1}{2}e^{\frac{t_0-T}{\varepsilon}} + \frac{1}{2}e^{\frac{2(t_0-T)}{\varepsilon}}\right] + T - t_0.$$

These results are summarized in the next proposition.

Proposition 2:

Given the system (29), there exists an optimal control $u_0^* \varepsilon U_c$ which minimizes the cost (25) subject to the constraint (26) for each appropriate set of initial conditions. This control is given by (31) and (32) where β and T are given as discussed in Lemmas 2 and 3.

It should be noticed that the singular perturbation comes into the problem (27) as ε approaches zero. This is clearly seen from

the expression of the optimal control (31) - (32), i.e.

$$\lim_{\varepsilon\to 0} k(\varepsilon) = \frac{1}{T-t_0}\left[x_3(t_0) - \beta + u_T(T-t_0)\right]$$

$$\lim_{\varepsilon\to 0} u_0^*(t,\varepsilon) = -\lim_{\varepsilon\to 0} k(\varepsilon)\left[e^{\frac{t-T}{\varepsilon}} - 1\right]$$

$$= u_T + \frac{x_3(t_0)-\beta}{T-t_0} \qquad t_0 \leq t < T,$$

which is exactly the same as that in (23). Therefore, for ε sufficiently small the solution to the optimal control problem (25) associated with a fourth-order system (27) can be approximated arbitrarily closely by the solution to the problem (8) associated with the third-order system (5).

In fact, not only the reduction of system order is shown here, but also an explicit solution to the optimization problem of the quadratic system† is derived. This provides a great deal of potential to implement such a control law in practice because the first-order actuator dynamics have been included. Actually, this result can be generalized to include any higher order actuator dynamics as long as the constancy of the control area is sustained.

---

† Note that (27) is no longer a bilinear system as defined in Section II; instead, it is sometimes referred to as a quadratic system.

V. A Least Squares Estimation

Given continuous measurements of the position coordinates $(x_1(t), x_2(t))$ and the missile turn rate $u(t)$ together with the proportionality parameter γ, a least squares estimate of v_T and $x_3(t_0)$ based on the knowledge of a constant angular maneuver of the target is considered in this section. The purpose in consideration of such a formulation is to provide an easily implemented estimator which connecting with the previously obtained optimal controller gives a feasible solution to the overall missile intercept problem.

Let the target speed v_T be modeled by a constant v, and let the initial relative heading $x_3(t_0)$ be denoted by "a". A least squares estimate of (a,v) results upon minimizing the functional:

$$\begin{aligned} J(a,v) = & \int_{t_0}^{T} \left\{\dot{x}_1(t) + v \sin[a+U(t)] - ux_2(t)\right\}^2 dt \\ & + \int_{t_0}^{T} \left\{\dot{x}_2(t) - v \cos[a+U(t)] + u[\gamma+x_1(t)]\right\}^2 dt \end{aligned} \tag{33}$$

where $U(t)$ is defined in terms of the known target and missile turn rate u_T and u by:

$$U(t) = \int_{t_0}^{t} (u_T-u)ds. \tag{34}$$

A necessary condition for the minimization of J is that the partial derivatives of J with respect to "a" and v vanish:

$$\left.\frac{\partial J}{\partial a}\right|_{(a^*,v^*)} = 0 \quad , \quad \left.\frac{\partial J}{\partial v}\right|_{(a^*,v^*)} = 0 \quad . \tag{35}$$

Observing that J is quadratic in v, the best estimate v^* can be uniquely determined in terms of a^*:

$$\begin{aligned} v^* = -\frac{1}{T-t_0}\Big\{ & x_1(T)\sin[a^*+U(T)] - x_1(t_0)\sin a^* + x_2(t_0)\cos a^* \\ & - x_2(T)\cos[a^*+U(T)] \\ & - \int_{t_0}^{T} u_T\Big[x_1(t)\cos[a^*+U(t)] + x_2(t)\sin[a^*+U(t)]\Big] + \gamma u \cos[a^*+U(t)]\,dt\Big\} \end{aligned} \tag{36}$$

Similarly,

$$0 = A\cos a^* + B\sin a^* \tag{37}$$

where

$$\begin{aligned} A = {} & x_1(T)\cos U(T) + x_2(T)\sin U(T) - x_1(t_0) \\ & + \int_{t_0}^{T}\Big\{[\gamma U(t)+u_T x_1(t)]\sin U(t) - u_T x_2(t)\cos U(t)\;dt\Big\} \end{aligned}$$

$$\begin{aligned} B = {} & -x_1(T)\sin U(T) + x_2(T)\cos U(T) - x_2(t_0) \\ & + \int_{t_0}^{T}\Big\{[\gamma U(t)+u_T x_1(t)]\cos U(t) + u_T x_2(t)\sin U(t)\;dt\Big\} . \end{aligned}$$

From Equation (37), we obtain

$$a^* = 2m\pi + \tan^{-1}\frac{A}{B} \quad , \qquad m = 0, \pm 1, \ldots \tag{38}$$

in which m is chosen so that $\left.\partial^2 J/\partial a^2\right|_{(a^*,v^*)} > 0$. This is equivalent to:

$$B \cos a^* > 0 \, . \tag{39}$$

After a^* is solved from (38) - (39), v^* is given by (36).

Simulation Results

The least squares estimation scheme is combined with the optimal control law to form a step-by-step control of the missile intercept system for selected initial conditions and target turn rates. The initial target turn rates are assumed to be 0.0 rad/sec and 0.15 rad/sec while the line speed is 1000 ft/sec. Two sets of sinusoidal fluctuations are imposed upon the target turn rate and line speed to test the sensitivity of the system. The analytical expressions of the target turn rate and line speed are listed in Table 1.

Table 1.

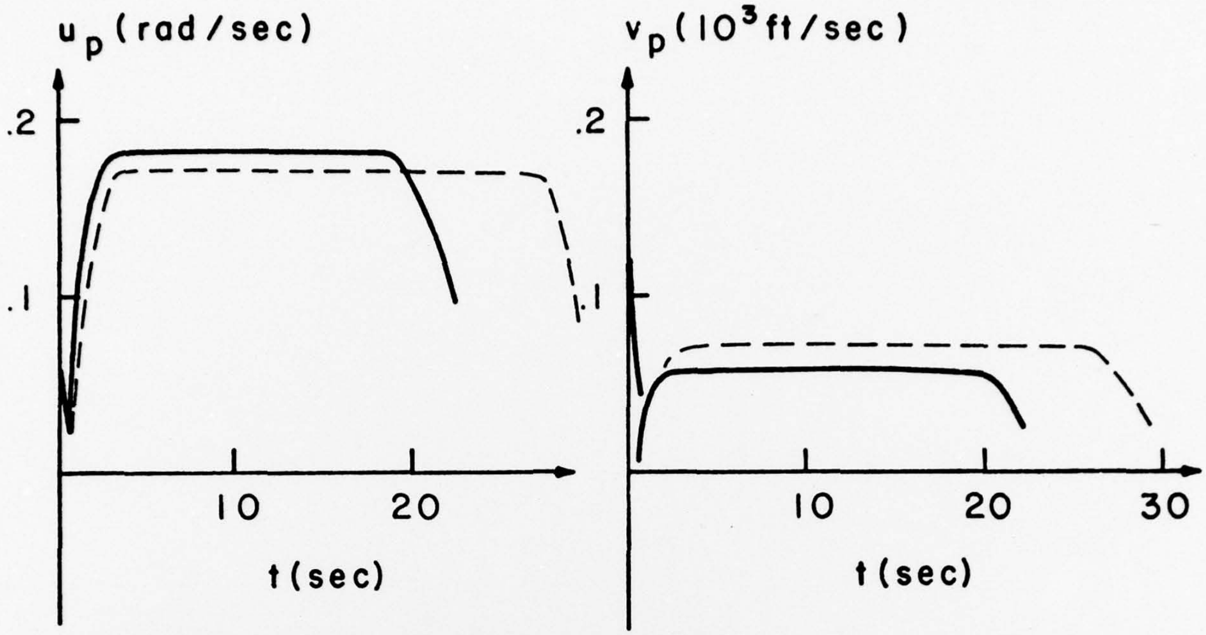
Actual Target Turn Rates and line speeds

Figure No.	u_T (rad/sec)	v_T (ft/sec)
2	0.15	1000
3	0.00	1000
4	-0.001+0.001(cos 0.2t+sin 0.2t)	1000
5	0.00	1000
6	0.00	980+20(cos 0.2t+sin 0.2t)

A time constant $\varepsilon = 0.5$ sec is assumed for the first-order actuator dynamics. During the first estimation interval of duration 0.5 sec., the control effort was assumed to be $u(t) = 0.06e^{-2t}$ rad/sec.

The relative trajectories of the target with respect to the missile are shown in Figures 2 - 6. It is clearly seen that in the case where either the target turn rate or its line speed is a known constant (Figures 2, 3 and 5), the estimation of the target speed and the initial relative heading is exact, and an ideal intercept is as expected. For the case in which an internal fluctuation of either the target turn rate or its line speed is not known in advance (Figures 4 and 6), the step-by-step control action is taken to offset the estimation error, which prolongs the expected intercept interval and in these cases results in reasonable convergence after a few loops of estimation.

Unknown fluctuations on the target turn rate of higher amplitude, e.g. $-0.005+0.005(\cos 0.2t+\sin 0.2t)$ were also considered, as well as a time-varying sawtooth target line speed, in order to test the effectiveness of the estimation scheme. Simulation results are relatively poor in these cases even though further estimation steps are called upon in trying to reduce the estimation error. This indicates the high sensitivity of the estimation scheme to derivations from the assumed constant values for the unknown parameters. It is suggested that a recursive estimator, which utilizes the first estimation data to initiate a secondary least squares estimation on either the target turn rate or line speed, may be satisfactory in this objective.



Trajs.	Initial Data		End Pt. Data	
	$\|X(0)\|$	$\beta(0)$	$\|X(T)\|$	$\beta(T)$
———	10000	57.29°	0	28.64°
- - - -	8060	57.29°	0	28.64°

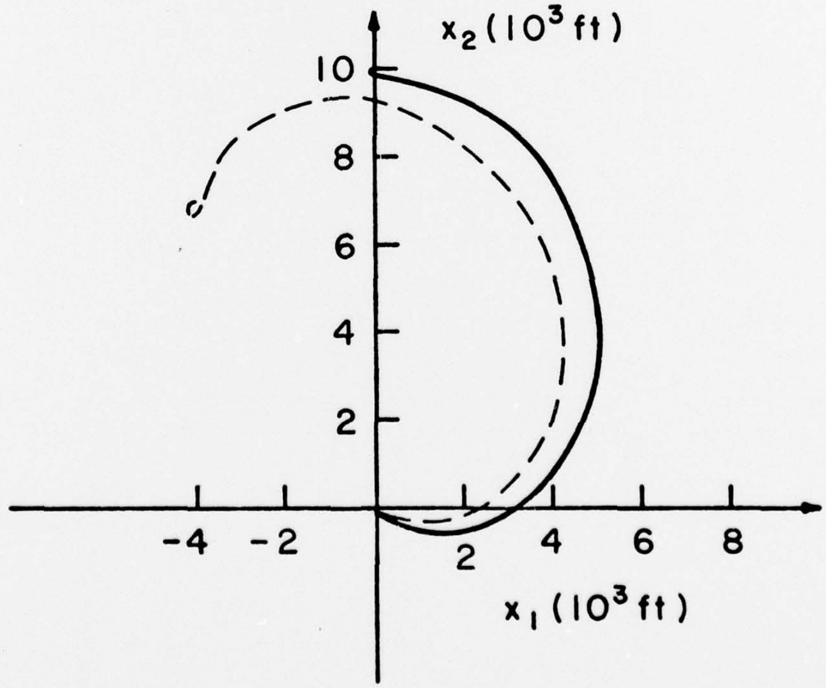
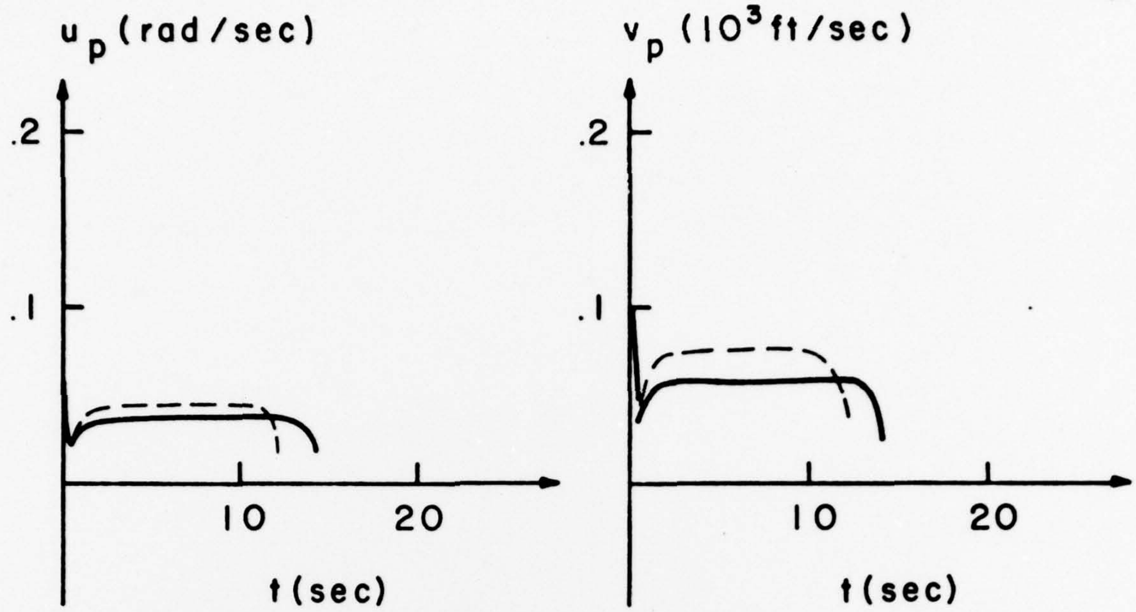


Figure 2. Missile Turn Rate, Speed and Relative Trajectories



Trajs.	Initial Data		End Pt. Data	
	$\|X(0)\|$	$\beta(0)$	$\|X(T)\|$	$\beta(T)$
———	10000	57.29°	0	28.64°
- - - -	8250	57.29°	0	28.64°

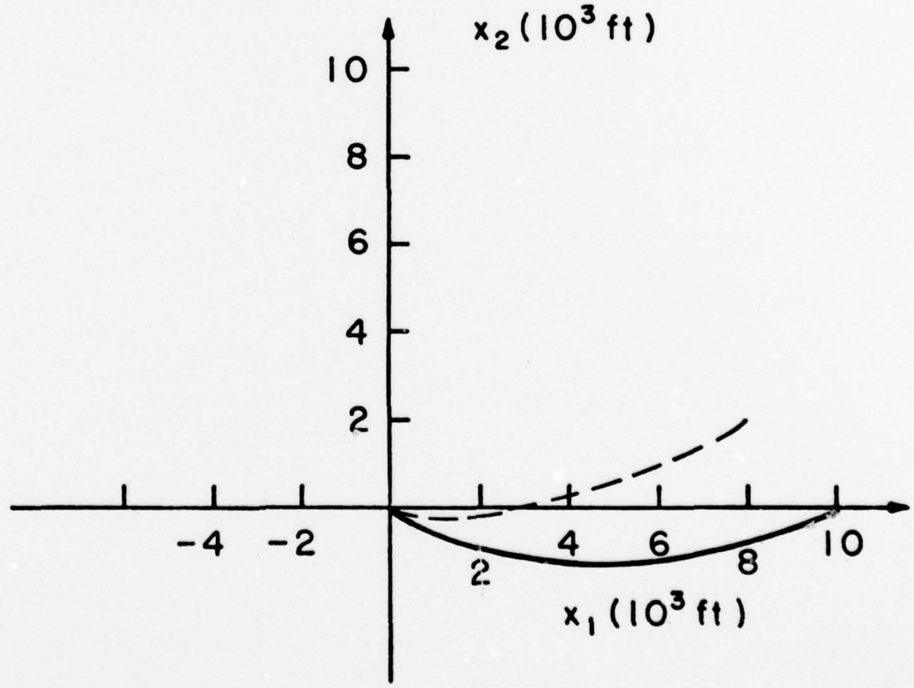
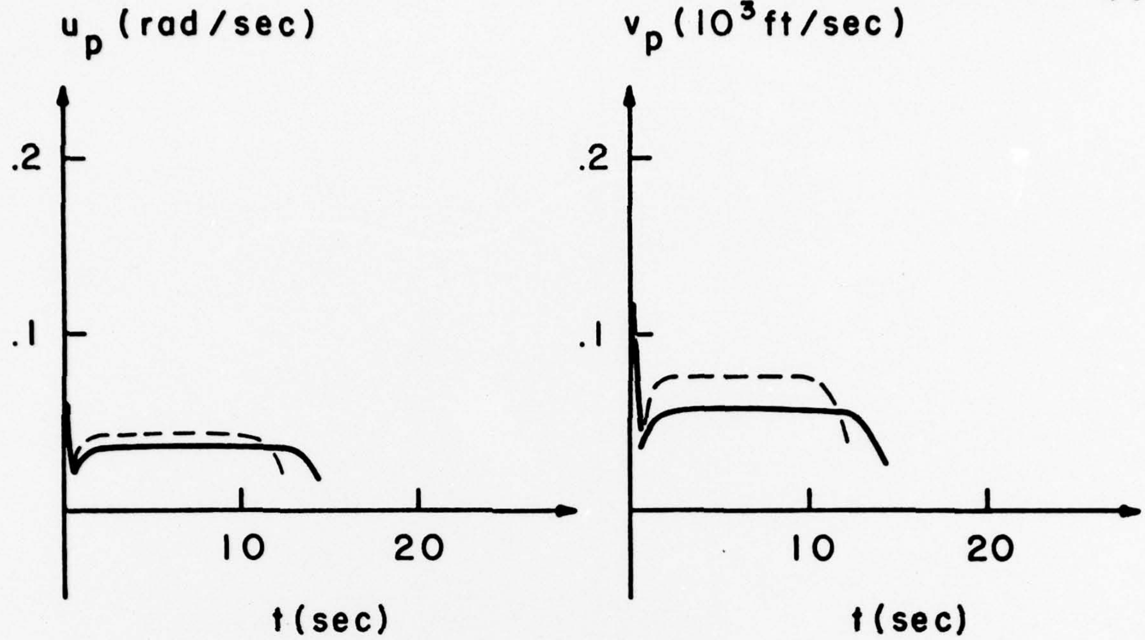


Figure 3. Missile Turn Rate, Speed and Relative Trajectories



Trajs.	Initial Data		End Pt. Data	
	$\|\|X(0)\|\|$	$\beta(0)$	$\|\|X(T)\|\|$	$\beta(T)$
———	10000	57.29°	11.5	28.5°
- - - -	8250	57.29°	14.5	28.6°

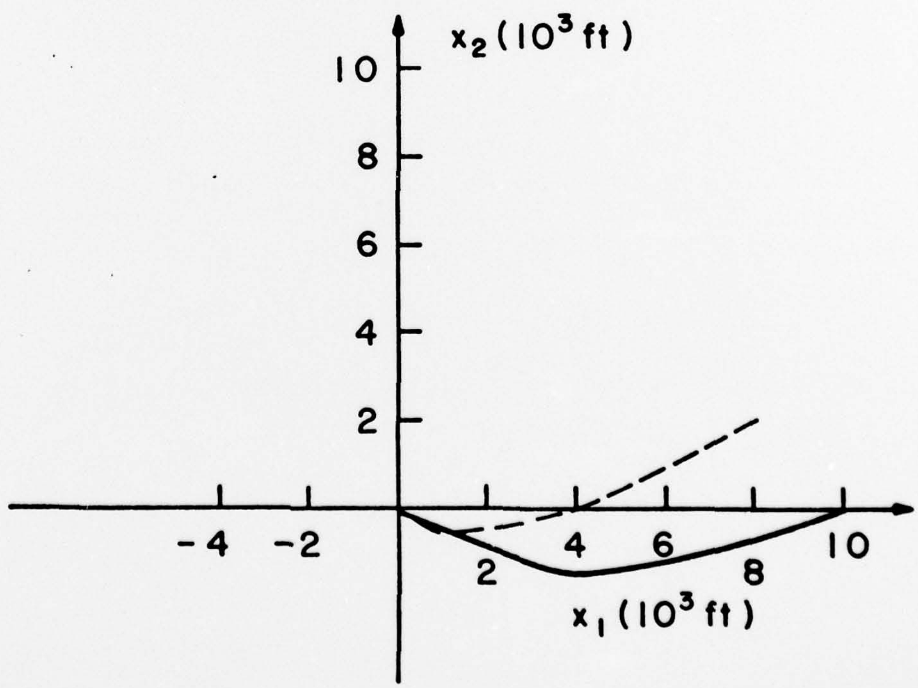
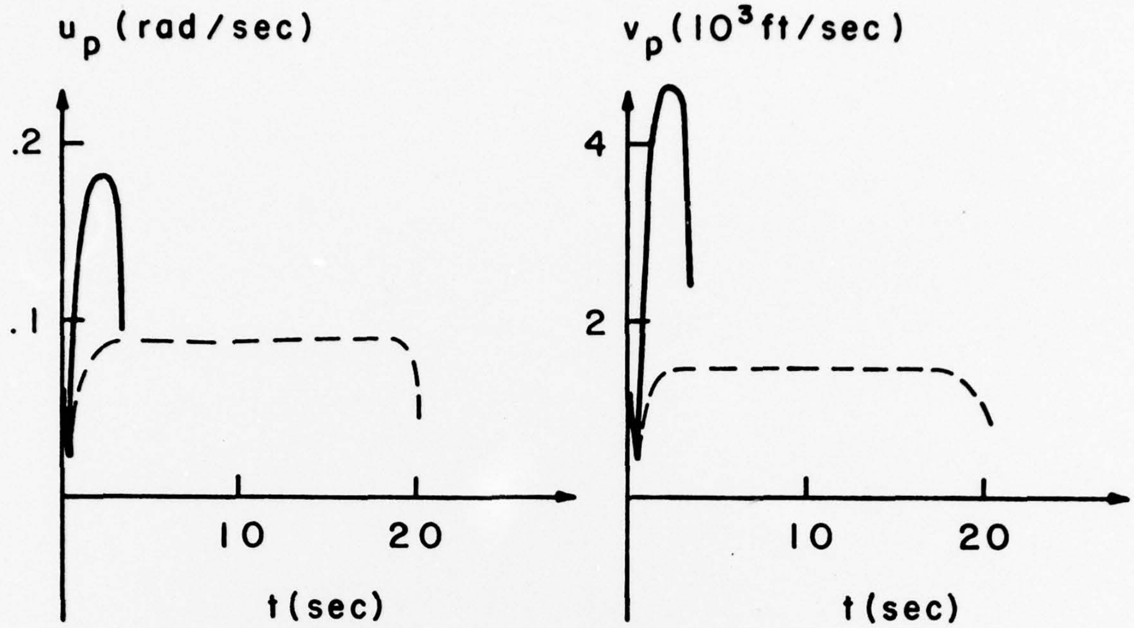


Figure 4. Missile Turn Rate, Speed and Relative Trajectories



Trajs.	Initial Data		End Pt. Data	
	$\|X(0)\|$	$\beta(0)$	$\|X(T)\|$	$\beta(T)$
———	10000	57.29°	6.56	28.63°
----	8060	57.29°	0	-41.88°

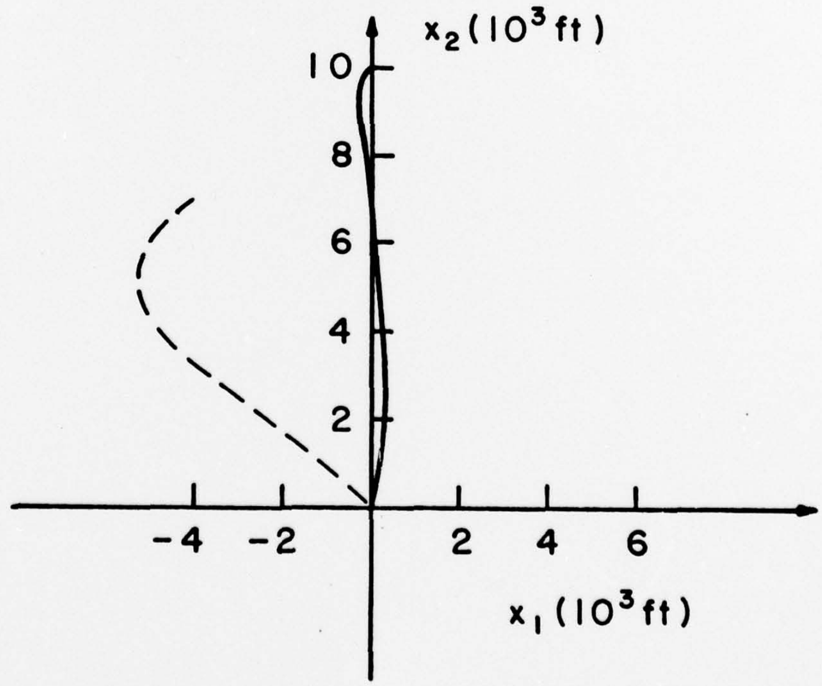
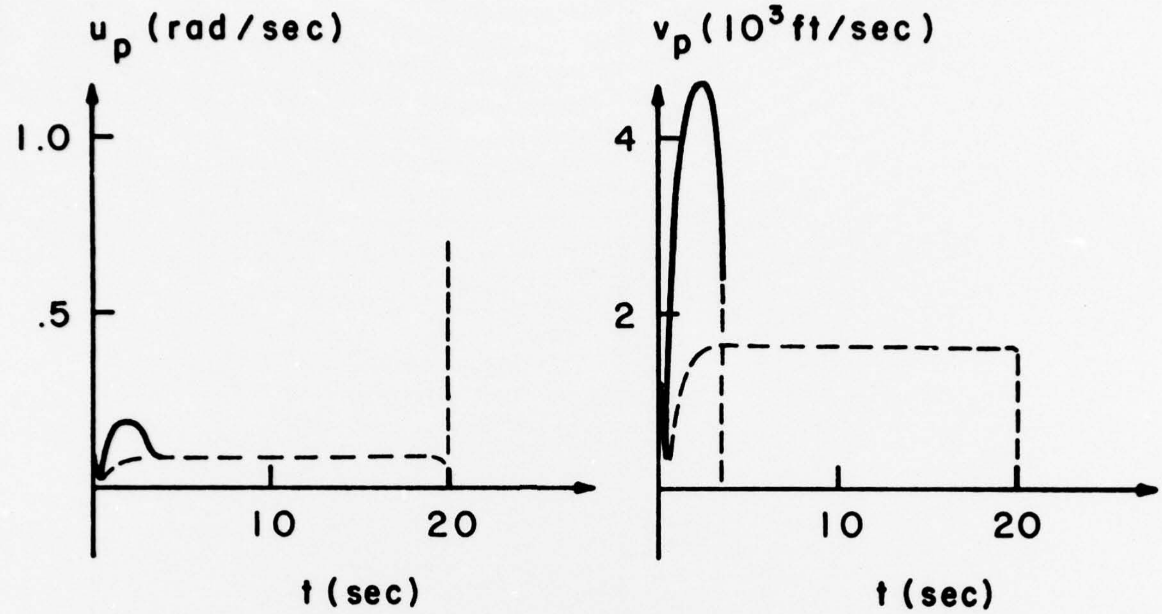


Figure 5. Missile Turn Rate, Speed and Relative Trajectories



Trajs.	Initial Data		End Pt. Data	
	$\lVert X(0) \rVert$	$\beta(0)$	$\lVert X(T) \rVert$	$\beta(T)$
———	10000	57.29°	10.32	28.60°
----	8060	57.29°	0.03	-54.03°

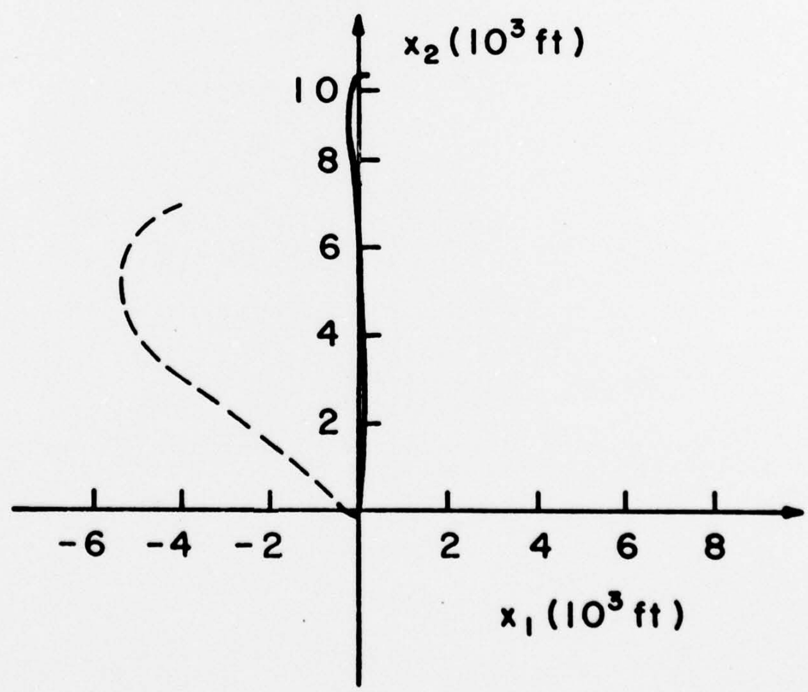


Figure 6. Missile Turn Rate, Speed and Relative Trajectories

VI. Conclusion

The application of the bilinear regulator theory to a two-dimensional pursuit-evasion problem was considered. An explicit solution to the minimum energy control of this problem was obtained by introducing a new design criterion that the line speed of the pursuer be made proportional to its turn rate. Optimal control appeared in a very simple form, i.e. a constant determined by the boundary conditions. On the other hand, it was shown that a sub-optimal control law can always be constructed which will drive the missile to the target at the same intercept time as long as the area swept out by this control function satisfies the same constant. This allows the control engineer a great deal of flexibility in the design of a feasible easily-implemented sub-optimal controller.

An extension of this problem to the singularly perturbed system was also studied. It was shown that a closed form solution for this higher-order system can also be obtained. This is in contrast with the asymptotically approximated solutions derived in Reference [7] for general nonlinear systems. A least squares estimate of the target speed and the relative heading was discussed. A straight-forward solution was obtained. This estimation scheme was combined with the optimal control law to form a step-by-step closed loop control law. The simulation results indicated good accuracy for an intercept when the estimation errors are small. However, the sensitivity analyses against different fluctuations on target line speed and turn rate reflect the necessity to incorporate a recursive estimation scheme in correcting the fluctuation effect, which is yet to be investigated.

Further studies on the estimation including the target turn rate and speed as stochastic models, as well as the construction of a feedback control law, should be considered for this problem.

REFERENCES

[1] Wei, K. C. and Pearson, A. E., "On the bilinear regulator problem with a pursuit-evasion application", (in preparation).

[2] Deyst, J. J. and Price, C. F., "Optimal stochastic guidance laws for tacticle missiles", AIAA J. of Spacecraft and Rockets, Vol. 10, No. 5., May 1973, pp. 301-308.

[3] Ho, Y.C., Bryson, A. E. and Baron, S., "Differential games and optimal pursuit evasion strategies", IEEE Trans. Automat. Contr., Vol. AC-10, No. 4, 1965, pp. 385-389.

[4] Rajan, N. and Prassad, V. R., "The pursuit-evasion problem of two aircrafts in a horizontal plane", Proc. of Conf. on Dec. and Contr., Houston, Texas, December 1975, pp. 635-636.

[5] Slater, G. L., "Applications of optimal control to the air-to-air tacticle missile problem", Math. Lab. Preprint Series No. 5, U.S. Air Force Aerospace Research Laboratories, Ohio, 1974.

[6] Nazaroff, G. J., "An optimal terminal guidance law", IEEE Trans. Automat. Contr. Vol AC-21, No. 3, June 1976, pp. 407-408.

[7] SINGULAR PERTURBATIONS:ORDER REDUCTION IN CONTROL SYSTEM DESIGN, American Society of Mechanical Engineers, New York 1972.

[8] Wei, K. C., "Optimal control of bilinear systems with some aerospace applications", Ph.D. dissertation, Brown University, Providence, R. I. 1976.